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Holographic Principle of Black Holes in Brans-Dicke Theory

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Brans type I black holes is a peculiar spherically symmetric solution found in geometrized gravity theories since the azimuthal factor of its horizon is divergent or vanishing under the classical approach of $r = 2M$. However, if we regard that the spherically symmetric solution is available only when all physical quantities of black holes are meaningful, then our investigation would be restricted to a special range of parameters and hence indicate a definite holographic relation to type-I black holes in Brans-Dicke theory. After that, we are able to investigate this holographic relation by making use of the brick wall method. Drawn a comparison between the arising result and a simulated entropy formula derived from the thermodynamical evolution, a variable cut-off factor α of Brans type I black holes is ultimately carried out.
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I. INTRODUCTION

In 1960's, Brans and Dicke developed a new relativistic theory of gravity[3, 4]. This theory can be regarded as an economic modification of general relativity which accommodates both Mach's principle and Dirac's large number hypothesis as new ingredients. It may be the most well-known alternative theory of classical gravity to Einstein's general relativity. Comparing with general relativity, there is a scalar field which can describe the gravity in Brans-Dicke theory, besides the metric of space-time which describes the geometry. As a beginning, we may write down the action of Brans-Dicke theory as follow

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}(\Phi R - \omega g^{\mu\nu}\frac{\nabla_\mu\Phi\nabla_\nu\Phi}{\Phi}) - \int d^4x\sqrt{-g}\cdot \mathrm{L}_m, \tag{1}$$

here $g^{\mu\nu}$ is the metric tensor; g is the determinant of the metric tensor; R is a curvature scalar of space-time; ω is the parameter of Brans-Dicke theory; Φ is a scalar field of gravity. The corresponding field equation reads

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{8\pi}{\Phi}T_{m\mu\nu} + \frac{\omega}{\Phi^2}(\Phi_{;\mu}\Phi_{;\nu} - \\ &\quad \frac{1}{2}g_{\mu\nu}\Phi_{;\rho}\Phi^{;\rho}) + \frac{1}{\Phi}(\Phi_{;\mu;\nu} - g_{\mu\nu}\Box^2\Phi), \qquad (2)\\ \Box^2\Phi &= \frac{8\pi}{3+2\omega}T_m{}^{\mu}{}_{\mu}. \qquad (3)\end{aligned}$$

Ever since the Brans-Dicke theory first appeared, four forms of the exact static spherically symmetric vacuum solutions have been found in [4]. Its Brans type-I black holes solution reads

$$\begin{aligned} ds^2 &= -(1-\frac{2M}{r})^{Q-\chi}dt^2 + (1-\frac{2M}{r})^{-Q}dr^2 + \qquad (4)\\ &\quad r^2(1-\frac{2M}{r})^{1-Q}d\Omega^2, \\ \Phi &= (1-\frac{2M}{r})^{\frac{\chi}{2}}. \qquad (5)\end{aligned}$$

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Here M is the mass of black holes, Q and χ are two numbers satisfying the following equation,

$$Q^2 + (1 + \frac{\omega}{2})\chi^2 - Q\chi - 1 = 0. \tag{6}$$

Obviously, Brans type-I black holes solution will be degenerated into a trivial solution —the Schwarzschild solution, when $Q \to 1$, $\chi \to 0$. In addition, there is one point in solution (4) should be of particular note. That is not only the radial factor of its horizon is divergent, but the azimuthal factor of its horizon is also divergent or vanishing under the classical approach of $r = 2M$. In history, there is controversy about the existence of nontrivial Brans-Dicke black holes. In [8], Hawking has firstly proved that any stationary black hole solution in Brans-Dicke theory, if it satisfies the weak energy condition, must be identical with that in general relativity. However, M.Campanelli and C.O.Lousto[6] have pointed out that in Brans-Dicke theory the nontrivial spherically symmetric solutions of black hole different from Schwarzschild solutions must be existent. For example, Brans type I solutions contain the nontrivial spherically symmetric solutions of Brans-Dicke theory. In [13], Kim has also argued that this fact was not conflict with Hawking Theorem, because these nontrivial Brans type I solutions violate the weak energy condition.

In addition, recent work[6, 16], especially for Kang began to propose a certain non-decreasing quantity defined on the event horizon which is proportional to the thermodynamical entropy of black holes in Brans-Dicke theory in [12], has heralded a renewed interest in Brans type I black holes. Kang's proposition reads

$$S_{bht} = \frac{1}{4}\oint_H d^2x\sqrt{h}\Phi(x). \tag{7}$$

Obviously, here the area-entropy relation is modified for black holes in Brans-Dicke theory. Note that the proof of above equation does not require the existence of regular event horizon like Hawking's area theorem. Kang accomplished the proof of the non decrease of this quantity under the assumption of the positivity of the scalar field Φ and the coupling constant ω in Brans-Dicke theory. The positivity of ω is natural since, otherwise, it gives unphysical negative energy matter in the theory. But for Φ, as far as we know, we have not found any physical system in the literature which shows the vanishing of Φ . Hence we suppose that this new formula of the entropy would be available to stationary black holes in Brans-Dicke theory. Before applying it to type I black holes solutions, we calculate the integral:

$$\begin{aligned} &\frac{1}{4}\oint_{r=r_h+\Delta} d^2x\sqrt{h}\Phi(x) \\ =\ &\pi \cdot (2M)^{1+Q-\frac{\chi}{2}} \cdot \Delta^{1-Q+\frac{\chi}{2}}. \end{aligned} \tag{8}$$

Obviously, the entropy vanishes under the classical approach to $r = 2M$. Therefore, we have to integrate above formula on the radius of $r = r_h + \Delta$ and Δ is a little quantity. When Δ is in limit to a zero, this integral can be identified to the thermodynamical entropy of Brans-Dicke black hole demonstrated in Eq.(8). Since an infinite or vanishing black hole entropy may be not desirable in physics, we have to restrict our discussion to the case of

$$1 - Q + \frac{\chi}{2} = 0. \tag{9}$$

Therefore, the non-vanishing entropy of Brans type I black hole may be written as:

$$\begin{aligned} S_{bht} &= \frac{1}{4}\oint_{r_h} d^2x\sqrt{h}\Phi(x) \\ &= \pi \cdot (2M)^{1+Q-\frac{\chi}{2}} \\ &= \pi \cdot (2M)^2. \end{aligned} \tag{10}$$

It is to say that a nontrivial Brans type I black hole satisfying the condition (9) is of the entropy equal to the Schwarzschild case. However, the entropy formula stated in Eq.(1) is derived from the simulation of the thermodynamics of Brans type I black holes. In theoretical, the entropy (1) of black holes must be inspected further by its corresponding microscopic statistical mechanism.

On the other hand, since the work of Bekenstein and Hawking [2, 9] our knowledge about black hole physics has improved quite considerably. Moreover, black hole physics is also the main gate towards understanding of gravity in extreme conditions, and as a consequence, of quantum gravity. This led t' Hooft and Susskind [10, 14] to generalize the area law relating entropy and the area of a black hole to any gravitational system by means of the introduction of the holographic principle, which in the last several years turned into a powerful means to the understanding of possible ways towards the quantization of gravity. This point motivate us to perform a deeper study of the thermodynamics of the black hole solution of Brans-Dicke theory. Meantime, the brick wall method [1, 10, 11] as a statistical approach to interpret the entropy of black holes has been studied much in Einstein's theory. For these reasons, we are also interested in investigating the holographic relation of Brans type I black holes on the level of its microscopic statistics by making use of the brick wall model.

II. BEKENSTEIN BOUND IN BRANS-DICKE GRAVITY

Within the framework of general relativity theory, Bekenstein[2] has proposed that there exists a universal upper bound to the entropy-to-energy ratio of any system of total energy E and effective proper radius α given by the inequality

$$S/E \leq 2\pi\alpha. \tag{11}$$

This bound has been checked in many physical situations, either for systems with maximal gravitational effects or systems with negligible self-gravity.

In this section we intend to consider how this bound behaves in Brans-Dicke gravity. Considering a neutral body of rest mass m, and proper radius α which is dropped into the Brans type I black hole, we demand that this process should satisfy the generalized second law(GSL).

Following Carter[7] and using the constants of motion

$$E = -\pi_t = -g_{tt}\dot{t}, \tag{12}$$
$$m = \sqrt{-g_{\alpha\beta}P^{\alpha}P^{\beta}}, \tag{13}$$

we get the equation of motion of the body on the background of (4)

$$E = m\sqrt{-g_{tt}}. \tag{14}$$

The energy at $r = r_h + \epsilon$ is given by

$$E = m(\frac{\epsilon}{2M})^{\frac{Q-\chi}{2}}. \tag{15}$$

In order to find the change in the black hole entropy caused by the assimilation of the body, one should evaluate E at the point of capture, a proper distance α outside the horizon

$$\begin{aligned} \alpha &= \int_{r_h}^{r_h+\epsilon} \sqrt{g_{rr}}dr \\ &= (2M)^{\frac{Q}{2}} \cdot \frac{\epsilon^{1-\frac{Q}{2}}}{1-\frac{Q}{2}}. \end{aligned} \tag{16}$$

The validity of the proper distance requires $2 > Q$. The assimilation of the body results in a change $dM = E$ in the black hole mass. Using the first law of thermodynamics

$$dM = TdS, \tag{17}$$

and the temperature relation

$$\begin{aligned} \kappa^2 &= -\frac{1}{2}(\nabla^a\xi^b)(\nabla_a\xi_b), \\ \kappa &= \frac{|Q-\chi|}{2}(1-\frac{2M}{r})^{Q-\frac{\chi}{2}-1} \cdot \frac{2M}{r^2}. \end{aligned} \tag{18}$$

We get the black hole entropy increases as

$$(dS)_{bh} = \frac{2-Q}{|Q-\chi|} \cdot 2\pi m\alpha. \tag{19}$$

However, we know according to GSL, that the relation $(\Delta S)_T \equiv (dS)_{bh} - S_{bo} \geq 0$ must be satisfied. This implies that the upper limit for the entropy of the body in Brans-Dicke gravity should be

$$S_{bo} \leq \frac{2-Q}{|Q-\chi|} \cdot 2\pi E\alpha. \tag{20}$$

III. STATISTICAL ENTROPIES

Next we check the holographic relation of this kind of black holes by making use of the brick wall method. Firstly, we consider a minimally coupled scalar field which satisfies the Klein-Gordon equation[10]

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)-m^2\phi=0. \tag{21}$$

The 't Hooft method consists in introducing a brick wall cut-off near the event horizon r_h, such that the boundary condition

$$\begin{aligned}\phi &= 0 \qquad for \qquad r\le r_h+\epsilon, &(22)\\ \phi &= 0 \qquad for \qquad r\ge L\gg r_h, &(23)\end{aligned}$$

is introduced. In the spherically symmetric space, the decomposition of the scalar field is given by

$$\phi(t,r,\theta,\varphi)=e^{-iEt}R(r)Y(\theta,\varphi). \tag{24}$$

Substituting this form into field equation (21) under the background of Brans-Dicke type-I black hole, we have

$$\begin{aligned}&E^2r^2(1-\frac{2M}{r})^{1-2Q+\chi}-m^2r^2(1-\frac{2M}{r})^{1-Q}\\ &+(1-\frac{2M}{r})^{\frac{\chi}{2}}\frac{1}{R}\frac{\partial}{\partial r}(r^2(1-\frac{2M}{r})^{1-\frac{\chi}{2}}\frac{\partial R}{\partial r})=\lambda.\end{aligned} \tag{25}$$

The eigenvalue equation of spherical function $Y(\theta,\varphi)$ has been well resolved as

$$\lambda=l(l+1) \qquad for \qquad l=0,1,2,3... \tag{26}$$

Using the WKB approximation, we substitute $\rho(r)e^{iS(r)}$ for $R(r)$, here the function $\rho(r)$ is a slowly varying amplitude and $S(r)$ is a rapidly varying phase. We obtain

$$\begin{aligned}&E^2r^2(1-\frac{2M}{r})^{1-2Q+\chi}-m^2r^2(1-\frac{2M}{r})^{1-Q}-\lambda\\ &+(1-\frac{2M}{r})^{\frac{\chi}{2}}(r^2(1-\frac{2M}{r})^{1-\frac{\chi}{2}})(-(\frac{\partial S}{\partial r})^2)\cong 0.\end{aligned} \tag{27}$$

Then we get the radial wave number $K\equiv\partial_r S$,

$$\begin{aligned}&K=(1-\frac{2M}{r})^{-Q+\frac{\chi}{2}}\cdot\\ &\sqrt{E^2-m^2(1-\frac{2M}{r})^{Q-\chi}-\frac{l(l+1)}{r^2}(1-\frac{2M}{r})^{-1+2Q-\chi}}.\end{aligned} \tag{28}$$

The number of radial modes n_r is given by

$$\pi n_r=\int_{r_h+\epsilon}^{L}drK(r,l,E). \tag{29}$$

For the partition function is

$$\begin{aligned}e^{-\beta F} &= \sum e^{-\beta E_{N_\tau}}=\sum_n e^{-\beta\sum_\tau n_\tau E_\tau}\\ &= \prod_\tau\sum_n e^{-\beta nE}\\ &= \prod_\tau(\frac{1}{1-e^{-\beta E}}),\end{aligned} \tag{30}$$

where E_{N_τ} is the total energy corresponding to the quantum state τ, and n_τ represents the set of quantum numbers associated to this system. The product $\prod$ takes the contribution from all the modes into account. So the free energy in this model is

$$F=\frac{1}{\beta}\int dlD_l\int dn_r\log(1-e^{-\beta E}). \tag{31}$$

Integrating by parts, we get

$$
\begin{aligned}
F &= -\int dl D_l \int dE n_r \frac{1}{e^{\beta E}-1} \\
&= -\frac{1}{\pi}\int dl(2l+1)\int dE \frac{1}{e^{\beta E}-1} \\
&\quad \cdot \int_{r_h+\epsilon}^{L} dr(1-\frac{2M}{r})^{-Q+\frac{\chi}{2}} \\
&\quad \cdot \sqrt{E^2 - m^2(1-\frac{2M}{r})^{Q-\chi} - \frac{l(l+1)}{r^2}(1-\frac{2M}{r})^{-1+2Q-\chi}}.
\end{aligned} \tag{32}
$$

We can regard the discrete variable $l(l+1)$ as a continuous one. Then we have

$$
\begin{aligned}
F &= \frac{1}{\pi}\int dE \frac{1}{e^{\beta E}-1}\int_{r_h+\epsilon}^{L} dr \cdot \\
&\quad (\frac{2}{3}m^2 r^2(1-\frac{2M}{r})^{1-Q} - \frac{2}{3}r^2E^2(1-\frac{2M}{r})^{1-2Q+\chi}) \cdot \\
&\quad (1-\frac{2M}{r})^{(-Q+\frac{\chi}{2})} \cdot \sqrt{E^2 - m^2(1-\frac{2M}{r})^{Q-\chi}}.
\end{aligned} \tag{33}
$$

To simplify our calculation, we set $m = 0$, and we have

$$
F = \frac{-2\pi^3}{45\beta^4}\int_{r_h+\epsilon}^{L} dr \cdot r^2(1-\frac{2M}{r})^{1-3Q+\frac{3}{2}\chi}, \tag{34}
$$

$$
\begin{aligned}
S &= \beta^2 \frac{\partial F}{\partial \beta} \\
&= \frac{8\pi^3}{45\beta^3}\int_{r_h+\epsilon}^{L} dr \cdot r^2(1-\frac{2M}{r})^{1-3Q+\frac{3}{2}\chi}.
\end{aligned} \tag{35}
$$

It is not difficult to find that the event horizon of type-I black hole is just $r_h = 2M$. According to Taylor expansion approximation we finally get the result as

$$
\begin{aligned}
S &\cong [\frac{1}{3}L^3 - M(1-3Q+\frac{3}{2}\chi)L^2 + 2M^2(1-3Q+\frac{3}{2}\chi)(-3Q+\frac{3}{2}\chi)L \\
&\quad -\frac{4}{3}M^3(1-3Q+\frac{3}{2}\chi)(-3Q+\frac{3}{2}\chi)\cdot(-1-3Q+\frac{3}{2}\chi)\ln L]\frac{8\pi^3}{45\beta^3} \\
&\quad -\frac{1}{(2M)^{(1-3Q+\frac{3}{2}\chi)}}[\frac{2-4(1-3Q+\frac{3}{2}\chi)-(1-3Q+\frac{3}{2}\chi)(-3Q+\frac{3}{2}\chi)}{2(4-3Q+\frac{3}{2}\chi)}\epsilon^{4-3Q+\frac{3}{2}\chi} + \\
&\quad \frac{2M(1+3Q-\frac{3}{2}\chi)}{3-3Q+\frac{3}{2}\chi}\epsilon^{3-3Q+\frac{3}{2}\chi} + \frac{4M^2}{2-3Q+\frac{3}{2}\chi}\epsilon^{2-3Q+\frac{3}{2}\chi}]\cdot\frac{8\pi^3}{45\beta^3}.
\end{aligned} \tag{36}
$$

Here the contribution from the next higher power of ϵ has been removed for that the brick wall method requires a small cut-off factor. So far as the brick wall method is available in our case, a finite number of the statistical entropy further requires that $-3Q + \frac{3}{2}\chi + 2 < 0$. Interesting, this requirement is automatically contained in the previously given parameter space (9). Furthermore, we can recall the definition of the surface gravity in (18) and a finite temperature ($\kappa = |Q-\chi| \cdot \frac{1}{4M}$) of Brans type I black holes is also available in the range of (9).

According to our knowledge of black hole thermodynamics, there should be a term in (36) which means the contribution of the black hole. The fact is just so. If we regard those terms which contain the infrared cutoff L as a contribution from the background, those terms which are not relevant to L should be attributed to black holes and their quantum correction. The coordinate distance ϵ has been transformed into the proper distance α in the equation (16). Furthermore, according to the definition of the surface gravity by Wald in [15], the surface gravity of type-I black holes (set $K_B = 1$) is given by the equation (18), then we have

$$
\beta^{-1} = |Q-\chi|(\frac{\epsilon}{r_h})^{Q-\frac{\chi}{2}-1}\cdot\frac{2M}{4\pi {r_h}^2}. \tag{37}
$$

The leading contribution of black holes entropy in (36) is of the lowest rank of ϵ, then we identify it with

$$\begin{aligned} S_{bhs} &= \frac{8\pi^3}{45\beta^3}\frac{-1}{(2M)^{(1-3Q+\frac{3}{2}\chi)}}\frac{4M^2}{2-3Q+\frac{3}{2}\chi}\epsilon^{2-3Q+\frac{3}{2}\chi} \\ &= \frac{1}{360}\frac{|Q-\chi|^3}{3Q-\frac{3\chi}{2}-2}\cdot(2M)^{\frac{2}{2-Q}}\cdot(\alpha(1-\frac{Q}{2}))^{\frac{2}{Q-2}}. \end{aligned} \tag{38}$$

It can also be verified that the statistical entropy formulation would degenerate into the case of Schwarzschild if $Q=1$ and $\chi=0$.

A physical area law (10), as the holographic principle of type I black holes of Brans-Dicke theory, has been has been found in the thermodynamic context. Comparing the result (38) derived from the brick wall method with the holographic relation of (10), we obtain a constraint on the proper distance of Brans type-I black holes

$$\alpha=[\frac{|Q-\chi|^3}{360\pi\cdot(3Q-\frac{3\chi}{2}-2)}]^{\frac{2-Q}{2}}\cdot(2M)^{Q-1}\cdot\frac{2}{2-Q}, \tag{39}$$

which can also be further checked by its Schwarzschild limit. After we set $Q=1$ and $\chi=0$, we can get $\pi=\frac{1}{360}\cdot\frac{4}{\alpha^2}$ from equation (39). It is consistent with the result of statistical interpretation of Schwarzschild black hole entropy derived in Einstein's theory. If such a statistical interpretation for nontrivial Brans type I black holes is valid, then the proper distance α is turned out to be a variable quantity. The running of α may attribute to the variability of Newton's constant G in Brans-Dicke theory. However, in above calculation of statistical entropy, we have assumed that the coordinate distance ϵ is a very small number in comparison with the radius of the horizon. We can check it according to the equations (39) and (16), and find that

$$\epsilon=[\frac{|Q-\chi|^3}{360\pi\cdot(3Q-\frac{3\chi}{2}-2)}]\cdot(2M)^{-1}. \tag{40}$$

This result implies that the brick wall method still works in the given range of parameters.

IV. CONCLUSION

A wide literature is present on the application of the brick-wall model to black holes solutions in Einstein's gravity theory for statistical interpretation. At the same time, the peculiarity of Brans type I black holes has also heralded a renewed interest in recent years. Therefore, this paper is aimed to check the holographic principle (10) of black holes in Brans-Dicke theory on the level of its microscopic statistics by making use of the brick wall method, and apply the arising interpretation to the thermodynamical simulated formula. Interestingly, it turns out that a variable proper distance α must be favored (39) in this case. A possible explanation for this point is that the Newton's constant G is also variable in Brans-Dicke theory and therefore at the end of the calculations always reflects this character.

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[1] Abdalla E and Alejandro Correa-Borbonet L ,2002,Phys.Rev. D**65** 124011, hep-th/0109129
[2] Bekenstein J D 1973 Phys Rev.D **7**,2333
[3] Brans C H and Dicke R H 1961 Phys Rev **124**,925
[4] Brans C H 1962 Phys Rev **125**,2194
[5] Buosso R 2002 Rev.Mod.Phys **74** 825-874
[6] Campanelli Mand ,Loust C O,1993 Int.J.Mod.Phys. D**2** 451-462, gr-qc/9301013.

[7] Carter B 1968 Phys. Rev **174**,1559
[8] Hawking S 1972 Commun. Math.Phys.**25** 167 .
[9] Hawking S, 1971 Phys Rev.Lett **26**; 1974 Nature **30** 248; 1975 Commun.Math.Phys **43** 199
[10] 't Hooft G, THU-93/26, gr-qc/9310026
[11] 't Hooft G 1985 Nucl.Phys.B**256**,727.
[12] Kang G ,1996,Phys.Rev. D**54** 7483-7489, gr-qc/9606020
[13] Kim H 1997 Nuovo Cim.B **112** 329
[14] Susskind, 1995, J.Math.Phys 36 6377.
[15] Wald R M 1984 General Relativity (Chicago University Press)
[16] Zaslavskii O B, 2002,Class.Quantum.Grav **19** 3783